Ellipticity conditions for the Lax operator of the KP equations

2000 J. Phys. A: Math. Gen. 33 7123

(http://iopscience.iop.org/0305-4470/33/40/309)

J. Phys. A: Math. Gen. **33** (2000) 7123–7135. Printed in the UK PII: S0305-4470(00)13175-0

Ellipticity conditions for the Lax operator of the KP equations

Giampiero Esposito†‡ and Boris G Konopelchenko§‖

† Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Complesso Universitario di Monte S Angelo, Via Cintia, Edificio N', 80126 Napoli, Italy

‡ Università di Napoli Federico II, Dipartimento di Scienze Fisiche, Complesso Universitario di Monte S Angelo, Via Cintia, Edificio N', 80126 Napoli, Italy

§ Dipartimento di Fisica, Università di Lecce, Via Arnesano, 73100 Lecce, Italy

‖ Istituto Nazionale di Fisica Nucleare, Sezione di Lecce, 73100, Italy

Received 7 April 2000

Abstract. The Lax pseudo-differential operator plays a key role in studying the general set of Kadomtsev–Petviashvili equations, although it is normally treated in a formal way, without worrying about a complete characterization of its mathematical properties. The aim of this paper is therefore to investigate the ellipticity condition. For this purpose, after a careful evaluation of the kernel with the associated symbol, the majorization ensuring ellipticity is studied in detail. This leads to non-trivial restrictions on the admissible set of potentials in the Lax operator. When their time evolution is also considered, the ellipticity conditions turn out to involve derivatives of the logarithm of the τ-function.

1. Introduction

Several important developments in modern mathematical physics are due to the investigation of pseudo-differential operators on $\mathbb{R}^m$ and on general Riemannian manifolds [1]. For our purposes, it is sufficient to recall the following basic properties.

(i) A linear partial differential operator P of order d can be written in the form

$$P \equiv \sum_{|\alpha|\leqslant d} a_\alpha(x) D_x^\alpha \tag{1.1}$$

where (here $\mathrm{i} \equiv \sqrt{-1}$)

$$|\alpha| \equiv \sum_{k=1}^{m} \alpha_k \tag{1.2}$$

$$D_x^\alpha \equiv (-\mathrm{i})^{|\alpha|} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m} \tag{1.3}$$

and a_α is a C^∞ function on $\mathbb{R}^m$ for all α. The associated *symbol* is, by definition,

$$p(x,\xi) \equiv \sum_{|\alpha|\leqslant d} a_\alpha(x)\xi^\alpha \tag{1.4}$$

i.e. it is obtained by replacing the differential operator D_x^α by the monomial ξ^α. The pair (x, ξ) may be viewed as defining a point of the cotangent bundle of $\mathbb{R}^m$, and the action of P on the elements of the Schwarz space $\mathcal{S}$ of smooth complex-valued functions on $\mathbb{R}^m$ of rapid decrease is given by

$$Pf(x) \equiv \int \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}\, p(x,\xi)\, f(y)\,\mathrm{d}y\,\mathrm{d}\xi \tag{1.5}$$

where the $dy = dy_1, \ldots, dy_m$ and $d\xi = d\xi_1, \ldots, d\xi_m$ orders of integration cannot be interchanged, since the integral is not absolutely convergent.

(ii) Pseudo-differential operators are instead a more general class of operators whose symbol need not be a polynomial but has suitable regularity properties. More precisely, let S^d be the set of all symbols $p(x, \xi)$ such that [1]

(1) p is smooth in (x, ξ), with compact x support.
(2) For all (α, β), there exist constants $C_{\alpha,\beta}$ for which

$$|D_x^\alpha D_\xi^\beta p(x, \xi)| \leqslant C_{\alpha,\beta}(1 + |\xi|)^{d-|\beta|} \tag{1.6}$$

for some *real* (not necessarily positive) value of d, where $|\beta| \equiv \sum_{k=1}^m \beta_k$ (see (1.2)). The associated *pseudo-differential operator*, defined on the Schwarz space and taking values in the set of smooth functions on $\mathbb{R}^m$ with compact support,

$$P : \mathcal{S} \to C_c^\infty(\mathbb{R}^m)$$

is defined in a way formally analogous to equation (1.5).

(iii) Let now U be an open subset with compact closure in $\mathbb{R}^m$, and consider an open subset U_1 whose closure $\overline{U}_1$ is properly included in U: $\overline{U}_1 \subset U$. If p is a symbol of order d on U, it is said to be *elliptic* on U_1 if there exists an open set U_2 which contains $\overline{U}_1$ and positive constants C_i so that

$$|p(x, \xi)|^{-1} \leqslant C_1(1 + |\xi|)^{-d} \tag{1.7}$$

for $|\xi| \geqslant C_0$ and $x \in U_2$, where

$$|\xi| \equiv \sqrt{g^{ab}(x)\xi_a\xi_b} = \sqrt{\sum_{k=1}^m \xi_k^2}. \tag{1.8}$$

The corresponding operator P is then elliptic.

From a mathematical point of view, pseudo-differential operators occur in many problems in global analysis [1, 2], and recent developments deal with the functional calculus of pseudo-differential boundary-value problems [3]. From a physical point of view, such a formalism is important in quantum gravity and quantum field theory [4–6]. In particular, we are here interested in an interdisciplinary field, i.e. a rigorous approach to the Kadomtsev–Petviashvili (hereafter KP) equations. Recall that the KP equation can be written in the form [7, 8]

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) = 3\alpha^2\frac{\partial^2 u}{\partial y^2}. \tag{1.9}$$

If $\alpha^2 = 1$, it describes an Hamiltonian wave system which is exactly solvable but not Liouville integrable. It exhibits a degenerative dispersion, the asymptotic states for $t \to \pm\infty$ do not coincide and an infinite number of invariants of motion exist. If $\alpha^2 = -1$, it describes an Hamiltonian wave system that is exactly solvable and completely integrable. It exhibits non-degenerative dispersion and lack of decay, and the asymptotic states for $t \to \pm\infty$ coincide upon imposing rapid-decrease boundary conditions. Moreover, the number of invariants of motion remains infinite [8].

The general set of KP equations may be described by using the first-order operator

$$T \equiv \frac{\partial}{\partial x} = \partial_x \tag{1.10}$$

with $x \in \mathbb{R}$, and the associated Lax pseudo-differential operator

$$L \equiv T + \sum_{k=1}^\infty u_k(x, t_1, t_2, t_3, \ldots)T^{-k} \tag{1.11}$$

where the functions u_k are here called the 'potentials'. By doing so, one allows in general for their dependence on an infinite number of time variables $t \equiv (t_1, t_2, \ldots, t_p, \ldots)$. On assuming that T^{-1} is a well defined inverse operator (see section 2), so that

$$T \cdot T^{-1} = T^{-1} \cdot T = \mathbb{I} \tag{1.12}$$

one can compose the Lax operator with itself, giving rise to its 'powers', i.e. $L^n \equiv L \cdot L^{n-1}$, for all $n = 2, 3, \ldots, \infty$. Each such power has a differential part, denoted by B_n. To begin one sets

$$B_1 \equiv T \tag{1.13}$$

and, by virtue of (1.12), one finds

$$B_2 \equiv T^2 + 2u_1 \tag{1.14}$$

$$B_3 \equiv T^3 + 3u_1 T + 3(u_2 + u_{1,x}) \tag{1.15}$$

and so on. The KP hierarchy of integrable equations is then defined by the generalized Lax equation [9, 10]

$$\frac{\partial L}{\partial t_n} = [B_n, L] = B_n L - L B_n \tag{1.16}$$

and by the Zakharov–Shabat equation

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} = [B_n, B_m] \tag{1.17}$$

which may be seen as the compatibility conditions of the linear equations

$$L\psi = \lambda\psi \tag{1.18}$$

and

$$\frac{\partial \psi}{\partial t_n} = B_n \psi \tag{1.19}$$

for all n, under the assumption that

$$\frac{\partial \lambda}{\partial t_n} = 0. \tag{1.20}$$

At this stage, since t_1 plays the same role as x, t_1 or x are used without distinction in the literature [10]. Once the equations (1.16) are written for all values of n, the coefficients of T^{-k} are equated, and this leads to an infinite set of equations

$$\frac{\partial u_k}{\partial t_n} = \varphi_{kn} \tag{1.21}$$

where φ_{kn} are certain differential polynomials in the potentials and their derivatives. For example, from the equations [10]

$$\frac{\partial u_1}{\partial t_2} = u_{1,xx} + 2u_{2,x} \tag{1.22}$$

$$\frac{\partial u_2}{\partial t_2} = u_{2,xx} + 2u_{3,x} + 2u_1 u_{1,x} \tag{1.23}$$

$$\frac{\partial u_1}{\partial t_3} = u_{1,xxx} + 3u_{2,xx} + 3u_{3,x} + 6u_1 u_{1,x} \tag{1.24}$$

one obtains the KP equation for u_1:

$$\frac{\partial}{\partial x}\left(4\frac{\partial u_1}{\partial t_3} - 12u_1\frac{\partial u_1}{\partial x} - \frac{\partial^3 u_1}{\partial x^3}\right) - 3\frac{\partial^2 u_1}{\partial t_2^2} = 0. \tag{1.25}$$

In section 2 we derive the kernel of the Lax pseudo-differential operator by using a Green function method and the theory of distributions when all potentials only depend on x. In section 3 we obtain the symbol from the kernel of section 2, and derive suitable majorizations which ensure ellipticity of the Lax operator. Strong ellipticity is studied in section 4. The behaviour of the ellipticity conditions under KP flows is investigated in section 5, and concluding remarks are presented in section 6, while the appendix describes relevant details.

2. The Lax operator and its kernel

Following [10], we first consider a 'restricted' form of the Lax operator, for which the potentials u_k only depend on x. The general form (1.11) will be restored in section 5, where time evolution is studied (cf section 3 of [10]).

Once the operator (1.10) is given, the inverse operator T^{-1} is an integral operator with kernel given by the Green function $G_1(x, y)$ of T. Its action on any function f in its domain reads

$$(T^{-1}f)(x) = \int_{-\infty}^{\infty} G_1(x, y) f(y)\, \mathrm{d}y \tag{2.1}$$

where the Green function G_1 obeys the equation

$$T_x G_1(x, y) = \delta(x, y). \tag{2.2}$$

More precisely, the Green function G_1 is a kernel which solves the equation

$$\frac{\partial}{\partial x} G_1(x, y) = 0 \qquad \forall x \neq y \tag{2.3}$$

and the jump condition [11]

$$\lim_{x\to y^+} G_1(x, y) - \lim_{x\to y^-} G_1(x, y) = 1. \tag{2.4}$$

The problem described by equations (2.3) and (2.4) is solved by

$$G_1(x, y) = A_{1,1}(y) \qquad \text{if} \quad x > y \tag{2.5a}$$

$$G_1(x, y) = A_{1,1}(y) - 1 \qquad \text{if} \quad x < y \tag{2.5b}$$

where $A_{1,1}(y)$ is an arbitrary smooth function of y unless a suitable boundary condition is specified (see below).

Similarly, the operators T^{-2}, T^{-3} and so on are integral operators with kernel given by the Green function of $T^2, T^3, \ldots,$ respectively. For example, the operator $T^2 = TT$ has a Green function $G_2(x, y)$ satisfying the differential equation

$$\frac{\partial^2}{\partial x^2} G_2(x, y) = 0 \qquad \forall x \neq y \tag{2.6}$$

the continuity condition

$$\lim_{x\to y^+} G_2(x, y) = \lim_{x\to y^-} G_2(x, y) \tag{2.7}$$

and the jump condition

$$\lim_{x\to y^+} \frac{\partial G_2}{\partial x} - \lim_{x\to y^-} \frac{\partial G_2}{\partial x} = 1. \tag{2.8}$$

Equations (2.6)–(2.8) are solved by

$$G_2(x, y) = A_{1,2}(y) + A_{2,2}(y)x \qquad \text{if} \quad x > y \tag{2.9a}$$

$$G_2(x, y) = y + A_{1,2}(y) + (A_{2,2}(y) - 1)x \qquad \text{if} \quad x < y \tag{2.9b}$$

where now two arbitrary functions $A_{1,2}$ and $A_{2,2}$ are involved because G_2 is the Green function of a second-order differential operator.

It is therefore clear that, assuming for the time being that u_k only depends on x, the Lax operator (1.11) can be viewed as an integral operator whose action is given by

$$(L\psi)(x) = \int_{-\infty}^{\infty} K(x, y)\psi(y)\,\mathrm{d}y \tag{2.10}$$

with kernel

$$K(x, y) = -\delta'(x, y) + \sum_{p=1}^{\infty} u_p(x)G_p(x, y) \tag{2.11}$$

where we have used the well known distributional action of the first derivative of the Dirac delta functional [12], and the Green function $G_p(x, y)$ can be expressed in the form

$$G_p(x, y) = \sum_{r=1}^{p} C_{r,p}(y)x^{r-1}. \tag{2.12}$$

The 'coefficients' $C_{r,p}$ are actually functions of y obeying a law of the type (see (2.5) and (2.9))

$$C_{r,p}(y) = A_{r,p}(y) \qquad \text{if} \quad x > y \tag{2.13a}$$

$$C_{r,p}(y) = B_{r,p}(y) \qquad \text{if} \quad x < y \tag{2.13b}$$

where the coefficients $B_{r,p}(y)$ in (2.13b) can be expressed in terms of the coefficients $A_{r,p}(y)$ after imposing the continuity conditions

$$\lim_{x\to y^+} \frac{\partial^q G_p}{\partial x^q} - \lim_{x\to y^-} \frac{\partial^q G_p}{\partial x^q} = 0 \qquad \forall q = 0, 1, \ldots, p-2 \tag{2.14}$$

and the jump condition

$$\lim_{x\to y^+} \frac{\partial^{p-1} G_p}{\partial x^{p-1}} - \lim_{x\to y^-} \frac{\partial^{p-1} G_p}{\partial x^{p-1}} = 1. \tag{2.15}$$

In [7], the Green function $G_1(x, y)$ given by (2.5a) and (2.5b) has been written in the form

$$G_1(x, y) = \tfrac{1}{2}[\theta(x - y) - \theta(y - x)] \tag{2.16}$$

where θ is the step function such that $\theta(x) = 1$ if $x > 0$, $\theta(0) = \frac{1}{2}$, $\theta(x) = 0$ if $x < 0$. Equation (2.16) corresponds to choosing

$$A_{1,1}(y) = \tfrac{1}{2} \tag{2.17}$$

in equations (2.5a) and (2.5b). The operator T^{-1} is then the integral operator

$$T^{-1} : f \to \tfrac{1}{2}\int_{-\infty}^{x} f(y)\,\mathrm{d}y - \tfrac{1}{2}\int_{x}^{\infty} f(y)\,\mathrm{d}y. \tag{2.18}$$

Similarly, the operator T^{-2} turns out to be the integral operator

$$\begin{aligned} T^{-2} : f \to{}& \tfrac{1}{4}\int_{-\infty}^{x} \mathrm{d}y \int_{-\infty}^{y} f(z)\,\mathrm{d}z - \tfrac{1}{4}\int_{-\infty}^{x} \mathrm{d}y \int_{y}^{\infty} f(z)\,\mathrm{d}z \\ & -\tfrac{1}{4}\int_{x}^{\infty} \mathrm{d}y \int_{-\infty}^{y} f(z)\,\mathrm{d}z + \tfrac{1}{4}\int_{x}^{\infty} \mathrm{d}y \int_{y}^{\infty} f(z)\,\mathrm{d}z. \end{aligned} \tag{2.19}$$

By comparison with (2.9a) and (2.9b) this leads to the evaluation of $A_{1,2}(y)$ and $A_{2,2}(y)$, and the procedure can be iterated (in principle) to obtain all $A_{r,p}(y)$ coefficients in (2.13a), while the $B_{r,p}(y)$ are obtained after imposing (2.14) and (2.15) as we said before. The details of the construction are indeed a little involved, and hence it is worth showing what can be done with the integral operator T^{-2} given in (2.19). On the one hand, equations (2.9*a*) and (2.9*b*) lead to

$$(T^{-2}f)(x) = \int_{-\infty}^{x} \mathrm{d}y\,[A_{1,2}(y) + xA_{2,2}(y)]f(y) + \int_{x}^{\infty} \mathrm{d}y\,[y + A_{1,2}(y) + x(A_{2,2}(y) - 1)]f(y). \tag{2.20}$$

On the other hand, by virtue of (2.19), $(T^{-2}f)(x)$ is also given by

$$(T^{-2}f)(x) = \int_{-\infty}^{x} \mathrm{d}y\,\tfrac{1}{4}h(y) + \int_{x}^{\infty} \mathrm{d}y\,(-\tfrac{1}{4}h(y)) \tag{2.21}$$

where

$$h(y) \equiv \int_{-\infty}^{y} f(z)\,\mathrm{d}z - \int_{y}^{\infty} f(z)\,\mathrm{d}z. \tag{2.22}$$

Direct comparison of the representations (2.20) and (2.21) of $(T^{-2}f)(x)$ therefore yields the equations

$$[A_{1,2}(y) + xA_{2,2}(y)]f(y) = \tfrac{1}{4}h(y) \tag{2.23}$$

$$[y + A_{1,2}(y) + x(A_{2,2}(y) - 1)]f(y) = -\tfrac{1}{4}h(y). \tag{2.24}$$

The addition of (2.23) and (2.24) leads to

$$[2A_{1,2}(y) + 2xA_{2,2}(y) + y - x]f(y) = 0 \tag{2.25}$$

which is satisfied for all $f(y)$ if and only if

$$2A_{1,2}(y) + y = 0 \tag{2.26}$$

$$(2A_{2,2}(y) - 1)x = 0. \tag{2.27}$$

Such a system is solved by

$$A_{1,2}(y) = -\frac{y}{2} \tag{2.28}$$

$$A_{2,2}(y) = \tfrac{1}{2} \tag{2.29}$$

which provides the desired explicit formula for the Green function $G_2(x, y)$, upon insertion into (2.9*a*) and (2.9*b*).

3. Symbol and ellipticity

Recall now that, if L is a pseudo-differential operator defined by a kernel K, this is related to the symbol $p(x, \xi)$ by the equation [3]

$$K(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi} p(x, \xi)\,\mathrm{d}\xi. \tag{3.1}$$

This equation can be inverted to give a very useful formula for the symbol, i.e. (cf equation (2.1.36) in [3])

$$p(x, \xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}z\cdot\xi} K(x, x - z)\,\mathrm{d}z. \tag{3.2}$$

Equation (3.2) is a key formula for our investigation, because the ellipticity of L is defined in terms of its symbol, as we know from the introduction, following [1].

In our problem, which involves $x \in \mathbb{R}$, the integral (3.2) reduces to

$$p(x,\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\xi} K(x, x-z)\,\mathrm{d}z \tag{3.3}$$

where the kernel $K(x, y)$ is expressed by (2.11)–(2.13), and we have to check that the inequality (1.7) is satisfied for $|\xi| \geqslant C_0$ to obtain ellipticity. Indeed, the symbol (3.3) turns out to be

$$p(x,\xi) = -\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\xi}\delta'(z)\,\mathrm{d}z + \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\xi} \sum_{p=1}^{\infty} u_p(x) \sum_{r=1}^{p} C_{r,p}(x-z)x^{r-1}\,\mathrm{d}z \tag{3.4}$$

and hence obeys the inequality

$$|p(x,\xi)| \geqslant \left| \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\xi}\delta'(z)\,\mathrm{d}z \right| - \left| \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\xi} \sum_{p=1}^{\infty} u_p(x) \sum_{r=1}^{p} C_{r,p}(x-z)x^{r-1}\,\mathrm{d}z \right|. \tag{3.5}$$

Of course, the first integral on the right-hand side of (3.5) becomes meaningful within the framework of Fourier transform of distributions [13]. In the simplest possible terms, one has actually to consider the parameter-dependent integral (here $a > 0$)

$$\begin{aligned} I_{1,a}(\xi) &\equiv \int_{-\infty}^{\infty} \mathrm{e}^{-az^2}\mathrm{e}^{-\mathrm{i}z\xi}\delta'(z)\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty} \delta(z)(-2az - \mathrm{i}\xi)\mathrm{e}^{-az^2-\mathrm{i}z\xi}\,\mathrm{d}z. \end{aligned} \tag{3.6}$$

By virtue of the property defining the Dirac delta functional, according to which [12]

$$(\delta, f) = f(0) \tag{3.7}$$

the integral (3.6) equals $-\mathrm{i}\xi$, and hence the first term on the right-hand side of (3.5) equals $|\xi|$. Now we distinguish two cases, depending on whether

$$f(x,\xi) \equiv |\xi| - \left| \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\xi} \sum_{p=1}^{\infty} u_p(x) \sum_{r=1}^{p} C_{r,p}(x-z)x^{r-1}\,\mathrm{d}z \right| \tag{3.8}$$

is positive or negative. If

$$f(x,\xi) > 0 \tag{3.9}$$

holds, the majorization (1.7) for the ellipticity of the restricted Lax operator is satisfied provided that, for $|\xi| \geqslant C_0$,

$$\left| \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\xi} \sum_{p=1}^{\infty} u_p(x) \sum_{r=1}^{p} C_{r,p}(x-z)x^{r-1}\,\mathrm{d}z \right| \leqslant |\xi| - C_1^{-1}(1+|\xi|)^d. \tag{3.10}$$

In contrast, if

$$f(x,\xi) < 0 \tag{3.11}$$

holds, the restricted Lax operator is elliptic provided that

$$\left| \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\xi} \sum_{p=1}^{\infty} u_p(x) \sum_{r=1}^{p} C_{r,p}(x-z)x^{r-1}\,\mathrm{d}z \right| \geqslant |\xi| - C_1^{-1}(1+|\xi|)^d \geqslant C_0 - C_1^{-1}(1+|\xi|)^d \tag{3.12}$$

for $|\xi| \geqslant C_0$. If the order d of the Lax operator is positive, we can further write that, for $|\xi| \geqslant C_0$, the majorization (3.10) becomes

$$\left| \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\xi} \sum_{p=1}^{\infty} u_p(x) \sum_{r=1}^{p} C_{r,p}(x-z)x^{r-1}\,\mathrm{d}z \right| \leqslant |\xi| - C_1^{-1}C_0^d. \tag{3.13}$$

If we are interested in sufficient conditions we can point out that, since the inequality (3.5) is always satisfied, whereas (1.7) only holds when L is elliptic, the sufficient condition for ellipticity of the restricted Lax operator is expressed by

$$C_1^{-1}(1+|\xi|)^d \leqslant |\xi| - \left| \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\xi} \sum_{p=1}^{\infty} u_p(x) \sum_{r=1}^{p} C_{r,p}(x-z)x^{r-1}\,\mathrm{d}z \right| \tag{3.14}$$

for $|\xi| \geqslant C_0$. This leads in turn to the inequality (3.10).

To sum up, if the function $f : T^*(\mathbb{R}) \rightarrow \mathbb{R}$ defined by (3.8) has no zeros (which is already a non-trivial requirement on the potentials u_p), the restricted Lax operator is elliptic provided that either (3.10) or (3.12) is satisfied. The majorization (3.10) is further simplified in the form (3.13) in the case of positive order of the Lax operator. A sufficient condition for ellipticity is given instead by (3.14), which coincides with (3.10). In particular, when $|\xi| = C_0$ and the equality sign is chosen in (3.14), the order d of the restricted Lax operator can be evaluated by the formula

$$d = \frac{\log(C_1(C_0 - I_\xi))}{\log(1+C_0)} \tag{3.15}$$

where

$$I_\xi \equiv \sup_{x\in\mathbb{R}} \left| \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\xi} \sum_{p=1}^{\infty} u_p(x) \sum_{r=1}^{p} C_{r,p}(x-z)x^{r-1}\,\mathrm{d}z \right| \tag{3.16}$$

bearing in mind that the modulus of ξ is equal to the constant C_0.

4. Strong ellipticity

In a thorough analysis of the ellipticity properties, strong ellipticity should also be studied. For this purpose, following [3], we assume that the symbol of the restricted Lax operator is *polyhomogeneous*, in that it admits an asymptotic expansion of the form

$$p(x,\xi) \sim \sum_{l=0}^{\infty} p_{d-l}(x,\xi) \tag{4.1}$$

where each term p_{d-l} has the homogeneity property

$$p_{d-l}(x,\gamma\xi) = \gamma^{d-l} p_{d-l}(x,\xi) \tag{4.2}$$

for $\gamma \geqslant 1$ and $|\xi| \geqslant 1$. The *principal symbol* p^0 of the Lax operator is then, by definition,

$$p^0(x,\xi) \equiv p_d(x,\xi). \tag{4.3}$$

Strong ellipticity is formulated in terms of the principal symbol, because it requires that

$$\mathrm{Re}\, p^0(x,\xi) = \tfrac{1}{2}[p^0(x,\xi) + p^0(x,\xi)^*] \geqslant c(x)|\xi|^d \tag{4.4}$$

where $x \in \mathbb{R}$, $c(x) > 0$ and $|\xi| \geqslant 1$. In other words, given a positive function c, the product $c(x)|\xi|^d$ should be always majorized by the real part of the principal symbol of the restricted Lax operator. Indeed, the symbol (3.4) is such that

$$p(x,\gamma\xi) = -\mathrm{i}\gamma\xi + \gamma^{-1} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\xi} \sum_{p=1}^{\infty} u_p(x) \sum_{r=1}^{p} C_{r,p} \left(x - \frac{z}{\gamma}\right) x^{r-1}\,\mathrm{d}z. \tag{4.5}$$

By virtue of (4.1), (4.2) and (4.5) we find that

$$-\mathrm{i}\gamma\xi + \gamma^{-1} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\xi} \sum_{p=1}^{\infty} u_p(x) \sum_{r=1}^{p} C_{r,p} \left(x - \frac{z}{\gamma}\right) x^{r-1}\,\mathrm{d}z \sim \sum_{l=0}^{\infty} \gamma^{d-l} p_{d-l}(x,\xi). \tag{4.6}$$

Moreover, the term on the right-hand side of (4.6) with $l = 0$ should be the one occurring in the condition (4.4) for strong ellipticity. A mathematical advantage of strong ellipticity lies in the possibility of having a well defined functional trace of the heat semigroup associated with the Lax operator [1, 3].

5. Behaviour of the ellipticity conditions under KP flows

To study the preservation (or violation) of the ellipticity conditions under KP flows one has to analyse the following problem: suppose that the conditions (3.10) or (3.12) are satisfied for $t = 0$. Are they still valid for all or some $t > 0$?

This means that we consider again the potentials u_k as in equation (1.11), i.e. as functions of x and t, where t is a concise notation for infinitely many time parameters $(t_1, t_2, \ldots)$. It should be stressed that we consider only one spatial variable and infinitely many time parameters, since otherwise it would be problematic, at least for the authors, to generalize formulae such as (3.2) aimed at obtaining the symbol from the kernel of the operator. For our purposes it is convenient to use formulae generating such potentials by means of a single function. This is made possible by the τ-function (see the appendix), because one finds [10]

$$u_1(x;t) = \frac{\partial^2}{\partial x^2} \log \tau \tag{5.1}$$

$$u_2(x;t) = \frac{1}{2}\left(-\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x \partial t_2}\right) \log \tau \tag{5.2}$$

$$u_3(x;t) = \frac{1}{6}\left(\frac{\partial^4}{\partial x^4} - 3\frac{\partial^3}{\partial x^2 \partial t_2} + 2\frac{\partial^2}{\partial x \partial t_3}\right) \log \tau - u_1^2 \tag{5.3}$$

and infinitely many other equations of the general form

$$u_k(x;t) = F_k(\log \tau) \tag{5.4}$$

where F_k is, in general, a nonlinear function of $\log \tau$. Equations (5.4), with k ranging from 1 to ∞, should be inserted into the ellipticity conditions (3.10) and (3.14), substituting therein u_k with $F_k(\log \tau)$ for all k. The resulting majorizations involve nonlinear functions of the logarithm of the τ-function.

Further progress can be made by considering a 'truncated' Lax operator, e.g.

$$\tilde{L} \equiv T + u_1(x;t)T^{-1} + u_2(x;t)T^{-2}. \tag{5.5}$$

This should not seem an arbitrary simplification, because the Lax operator is obtained from the W operator of the appendix as [10]

$$L \equiv WTW^{-1}. \tag{5.6}$$

Now both W and its 'truncated version' [10]

$$W_m \equiv 1 + \sum_{k=1}^{m} w_k T^{-k} \tag{5.7}$$

satisfy the Sato equation (A.8), from which the generalized Lax equation (1.16) is eventually obtained. Thus, operators such as $\tilde{L}$ in (5.5) can be obtained from (5.6) if W is replaced by W_m therein. In this case, on defining

$$\begin{aligned} F(x,z;t) &\equiv \sum_{p=1}^{2} u_p(x;t) \sum_{r=1}^{p} C_{r,p}(x-z)x^{r-1} \\ &= u_1(x;t)C_{1,1}(x-z) + u_2(x;t)[C_{1,2}(x-z) + xC_{2,2}(x-z)] \end{aligned} \tag{5.8}$$

the integral on the right-hand side of the ellipticity condition (3.14) reduces to $(J_1 + J_2 + J_3)(x, \xi; t)$, where, bearing in mind that (see (2.17), (2.28) and (2.29))

$$C_{1,1}(x-z) = C_{2,2}(x-z) = \tfrac{1}{2} \quad \text{if} \quad z > 0, \quad -\tfrac{1}{2} \quad \text{if} \quad z < 0 \tag{5.9}$$

$$C_{1,2}(x-z) = -\frac{x}{2} + \frac{z}{2} \quad \text{if} \quad z > 0, \quad \frac{x}{2} - \frac{z}{2} \quad \text{if} \quad z < 0 \tag{5.10}$$

one finds

$$J_1(x, \xi; t) \equiv \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\xi} u_1(x; t) C_{1,1}(x-z)\, \mathrm{d}z = u_1(x; t) \lim_{b\to\infty} \left(\frac{2}{\xi} \mathrm{e}^{-\mathrm{i}b\frac{\xi}{2}} \sin \frac{b\xi}{2} \right) \tag{5.11}$$

$$\begin{aligned} J_2(x, \xi; t) &\equiv \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\xi} u_2(x; t) C_{1,2}(x-z)\, \mathrm{d}z \\ &= -x u_2(x; t) \frac{J_1(x, \xi; t)}{u_1(x; t)} + u_2(x; t) \mathrm{i} \frac{\mathrm{d}}{\mathrm{d}\xi} \frac{J_1(x, \xi; t)}{u_1(x; t)} \end{aligned} \tag{5.12}$$

$$J_3(x, \xi; t) \equiv \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\xi} u_2(x; t) x C_{2,2}(x-z)\, \mathrm{d}z = x u_2(x; t) \frac{J_1(x, \xi; t)}{u_1(x; t)}. \tag{5.13}$$

In these equations, the infinite upper limit of integration can be recovered by taking the limit as $b \to \infty$ of integrals from 0 to b (the lower limit being amenable to 0 by virtue of (5.9) and (5.10)). However, divergences remain, and hence we are only able to obtain well defined formulae by integrating up to finite values of b. The cancellation of terms involving $x u_2(x; t)$ is thus found to occur on performing the sum, and our integral reads eventually

$$(J_1 + J_2 + J_3)_b(x, \xi; t) = u_1(x; t) F_b(\xi) + u_2(x; t) \mathrm{i} \frac{\mathrm{d}}{\mathrm{d}\xi} F_b(\xi) \tag{5.14}$$

having defined

$$F_b(\xi) \equiv \frac{2}{\xi} \mathrm{e}^{-\mathrm{i}b\frac{\xi}{2}} \sin \frac{b\xi}{2}. \tag{5.15}$$

Now we point out that

$$\begin{aligned} |u_1(x; t) F_b(\xi) + u_2(x; t) \mathrm{i} F_b'(\xi)| &\leqslant |u_1(x; t) F_b(\xi)| + |u_2(x; t) \mathrm{i} F_b'(\xi)| \\ &\leqslant \frac{2}{\xi} |u_1(x; t)| + \left(\frac{b}{\xi} + \frac{2}{\xi^2} \right) |u_2(x; t)|. \end{aligned} \tag{5.16}$$

Thus, a sufficient condition for the validity of the majorization (3.14) is expressed by

$$|\xi| \geqslant C_1^{-1} (1 + |\xi|)^d + \frac{2}{\xi} |u_1(x; t)| + \left(\frac{b}{\xi} + \frac{2}{\xi^2} \right) |u_2(x; t)|. \tag{5.17}$$

It should be stressed that divergent integrals occur already in the ‘time-independent’ ellipticity condition (3.14). When the time parameters are introduced, it may be easier or harder to fulfill ellipticity, depending on the behaviour of $u_1, u_2, \ldots$ (which, in turn, all depend on the τ-function).

6. Concluding remarks

Our paper has been motivated by the need to obtain a deeper understanding of the basic structures of modern nonlinear physics. It is indeed well known that the Sato equation (A.8) generates the generalized Lax equation (1.16), the Zakharov–Shabat equation (1.17) and the inverse spectral transform scheme [10]. Moreover, an infinite number of nonlinear evolution equations (i.e. the KP hierarchy), of which the KP equation is the simplest nontrivial one, share solutions, and the τ-function makes it possible to express all such solutions.

The work in this paper is the first step towards a rigorous investigation of the ellipticity properties of the Lax pseudo-differential operator. We have found that, to achieve ellipticity, including its strong form, the various potentials $u_k(x)$ are no longer arbitrary, but should be chosen in such a way that the following conditions hold.

(i) The function $f : T^*\mathbb{R} \to \mathbb{R}$ defined in (3.8) has no zeros.
(ii) The majorization (3.14) holds (some care is actually necessary to deal with the integrals on the right-hand side of (3.14), as is clear from the analysis performed in section 5).
(iii) The asymptotic expansion (4.6) can be obtained. Note, however, that violation of (4.2) (i.e. lack of homogeneity) for $|\xi| < 1$ can cause logarithmic terms in the asymptotic expansion of the kernel defined by (2.11)–(2.13) (cf [14]).

Moreover, on allowing for the time evolution of the potentials in the Lax operator, now viewed as functions of x and of infinitely many time variables, we have obtained an explicit ellipticity condition in terms of the τ-function, when attention is restricted to the 'truncated' Lax operator (5.5). It now remains to be seen how to deal with the infinite sum over p in the ellipticity condition (3.14) when the potentials $u_p(x; t)$ appropriate for the 'full' Lax operator (1.11) are instead considered. The form of u_1 and u_2 remains the one given in (5.1) and (5.2), but the occurrence of an infinite number of such potentials makes it hard to re-express the time-dependent form of the majorization (3.14). The work in [15] has indeed obtained a very useful formula for the τ-function, but this remains of little help when the infinite sum over all potentials is performed.

A further interesting issue is the investigation of ellipticity for the formulation of KP hierarchy considered in [16], where the main new technique, when compared to the traditional approach to the generalized Lax equation, consists of replacing the Lax operator by an nth-order formal pseudo-differential operator

$$L_n \equiv T^n + \sum_{j=-\infty}^{n-2} q_j T^j \qquad n \geqslant 2. \tag{6.1}$$

The authors of [16] have been able to factorize L_n into $n-1$ first-order formal differential operators A_k, $1 \leqslant k \leqslant n-1$, and one first-order formal pseudo-differential operator $\tilde{A}_n$, i.e.

$$L_n = \tilde{A}_n A_{n-1}, \ldots, A_2 A_1 \tag{6.2}$$

where

$$A_k \equiv T + \eta_{k,x} \qquad 1 \leqslant k \leqslant n \tag{6.3}$$

$$\sum_{k=1}^{n} \eta_{k,x} = 0 \tag{6.4}$$

$$\tilde{A}_n \equiv A_n + \sum_{j=-\infty}^{-1} b_{n,j} T^j. \tag{6.5}$$

The results and unsolved problems described so far seem to show that new exciting developments might be obtained from the effort of combining some key techniques of nonlinear physics and the tools of linear and pseudo-differential operator theory. In particular, the mathematical requirement of ellipticity in the various forms considered in sections 3–5 restricts the potentials u_k in a form not previously considered in the literature to our knowledge, which might be used to select the realizations of the Lax operator one is interested in.

Acknowledgments

This paper has been partially supported by PRIN97 'Sintesi'. Correspondence with Gerd Grubb has been helpful for the authors, as well as constant encouragement from Giuseppe Marmo. Giampiero Esposito dedicates the present work to his daughter Margherita.

Appendix

Since the general reader is not necessarily familiar with the theory of τ-functions, we summarize the main properties hereafter. Following [10], we study for the operator W_m defined in (5.7) the ordinary differential equation

$$W_m \partial^m f(x) = (\partial^m + w_1(x)\partial^{m-1} + \cdots + w_m(x)) f(x) = 0 \tag{A.1}$$

which has m linearly independent solutions $f^{(1)}(x), \ldots, f^{(m)}(x)$. On writing equation (A.1) m times with $f = f^{(1)}(x), \ldots, f = f^{(m)}(x)$, one finds a linear system which can be solved for $w_j(x)$, for all $j = 1, \ldots, m$, and hence W_m is found from the definition (5.7).

When the w_j are assumed to depend also on infinitely many time variables $(t_1, \ldots, t_p, \ldots)$, the operator $W_m(x; t)$ is found to be

$$W_m(x; t) = \frac{\det A}{\tau(x; t)} \tag{A.2}$$

where, given the functions $h_0^{(j)}(x; t)$ and $h_n^{(j)}(x; t)$ such that

$$\left(\frac{\partial}{\partial t_n} - \frac{\partial^n}{\partial x^n}\right) h_0^{(j)}(x; t) = 0 \qquad n = 1, 2, \ldots \tag{A.3}$$

$$h_0^{(j)}(x; 0) = f^{(j)}(x) \tag{A.4}$$

$$h_n^{(j)}(x; t) = \frac{\partial}{\partial t_n} h_0^{(j)}(x; t) = \frac{\partial^n}{\partial x^n} h_0^{(j)}(x; t) \tag{A.5}$$

one has (see the definition (1.10))

$$A \equiv \begin{pmatrix} h_0^{(1)} & \cdots & h_0^{(m)} & T^{-m} \\ \cdots & \cdots & \cdots & \cdots \\ h_{m-1}^{(1)} & \cdots & h_{m-1}^{(m)} & T^{-1} \\ h_m^{(1)} & \cdots & h_m^{(m)} & 1 \end{pmatrix} \tag{A.6}$$

and

$$\tau(x; t) \equiv \det \begin{pmatrix} h_0^{(1)} & \cdots & h_0^{(m)} \\ \cdots & \cdots & \cdots \\ h_{m-1}^{(1)} & \cdots & h_{m-1}^{(m)} \end{pmatrix}. \tag{A.7}$$

The function τ is the τ-function used in equations (5.1)–(5.4), and the time evolution of the operator $W_m(x; t)$ is determined by the Sato equation

$$\frac{\partial W_m}{\partial t_n} = B_n W_m - W_m T^n \tag{A.8}$$

where the operators B_n are the same as occur in the generalized Lax equation (1.16).

References

[1] Gilkey P B 1995 *Invariance Theory, The Heat Equation and The Atiyah–Singer Index Theorem* (Boca Raton, FL: Chemical Rubber Company)

[2] Seeley R T 1969 *Topics in Pseudo-Differential Operators: CIME Conf. on Pseudo-Differential Operators* ed L Nirenberg (Roma: Edizioni Cremonese)
[3] Grubb G 1996 *Functional Calculus of Pseudodifferential Boundary Problems (Progress of Mathematics vol 65)* (Boston: Birkhäuser)
[4] Esposito G 1999 *Class. Quantum Grav.* **16** 1113
[5] Esposito G 1999 *Class. Quantum Grav.* **16** 3999
[6] Esposito G and Stornaiolo C 2000 *Int. J. Mod. Phys.* A **15** 449
[7] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)
[8] Zakharov V E 1991 *What is Integrability?* (Berlin: Springer)
[9] Date E, Kashiwara M, Jimbo M and Miwa T 1983 Transformation groups for soliton equations *Proc. RIMS Symp. on Nonlinear Integrable Systems, Classical Theory and Quantum Theory* ed M Jimbo and T Miwa (Singapore: World Scientific)
[10] Ohta Y, Satsuma J, Takahashi D and Tokihiro T 1988 *Prog. Theor. Phys.* (Suppl.) **94** 210
[11] Lanczos C 1961 *Linear Differential Operators* (London: Van Nostrand–Reinhold)
[12] Roos B W 1969 *Analytic Functions and Distributions in Physics and Engineering* (New York: Wiley)
[13] Kolmogorov A N and Fomin S V 1980 *Elements of the Theory of Functions and Functional Analysis* (Moscow: Mir)
[14] Grubb G and Seeley R T 1995 *Inv. Math.* **121** 481
[15] Chau L L, Shaw J C and Yen H C 1992 *Commun. Math. Phys.* **149** 263
[16] Gesztesy F and Unterkofler K 1995 *Diff. Int. Eqn* **8** 797